

LETTER TO THE EDITOR

Directed percolation and Reggeon field theory

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Abstract. Directed bond percolation is shown to be in the same universality class as Reggeon field theory. The critical behaviour and critical exponents near the percolation threshold are thereby inferred.

In the problem of directed bond percolation (Obukhov 1980), a preferred direction (labelled, say, by a coordinate t) is chosen, and the bonds are oriented with respect to this direction. The probability that a given bond is present depends on its orientation, and percolation is allowed only in the direction of orientation of a given bond. In the extreme case which we shall consider, only those bonds oriented in the direction of increasing t are allowed, with probability p . The problem then corresponds to a Markov process, since the probability $P(\{x_i\}, t)$ that a given set of sites (x_i, t) are connected to a given initial site depends in a straightforward way on the probabilities $P(\{x_j\}, t-1)$. If we regard the variable t as time, the allowed configurations of bonds will represent the time development of a random walk in x space in which branching, recombination and absorption can occur. Such models arise in chemistry and biology (Schlögl 1972). It was pointed out by Grassberger and Sundermeyer (1978) and Grassberger and de la Torre (1979) that such models are in the same universality class as Reggeon field theory (RFT) (Abarbanel *et al* 1975b, Moshe 1978). This is a theory of the scattering of elementary particles at high energies and low-momentum transfers, in which t has the interpretation of the logarithm of the longitudinal momentum (rapidity) and x is the impact parameter. The correlation functions are related to scattering cross sections.

In this Letter, we establish directly the connection between directed percolation and RFT. Since much work has been done on the latter, we can immediately deduce many results on the critical behaviour of the former.

Since we are interested in calculating the probability $G(x_2, t_2; x_1, t_1)$ of percolation from (x_1, t_1) to (x_2, t_2) ($t_1 < t_2$) irrespective of the other sites, a simple set of diagrammatic rules suffices. (1) Place bonds on the lattice in such a way that every bond is connected by other bonds to (x_1, t_1) always in the direction of decreasing t , and to (x_2, t_2) in the direction of increasing t . (2) Insert a factor p for each bond. (3) Insert a factor (-1) for each closed loop.

G is then the sum over all such terms. The last factor is to avoid multiple counting. For example, in figure 1 the diagram (a) counts all configurations where bonds AB_1 and B_1C are present, irrespective of whether AB_2 and B_2C are present. Similarly for (b). Thus the diagram (c) must be subtracted off to avoid double counting. This rule can be replaced by: (3)' Insert a factor $-i^n$ for each vertex where n bonds meet.

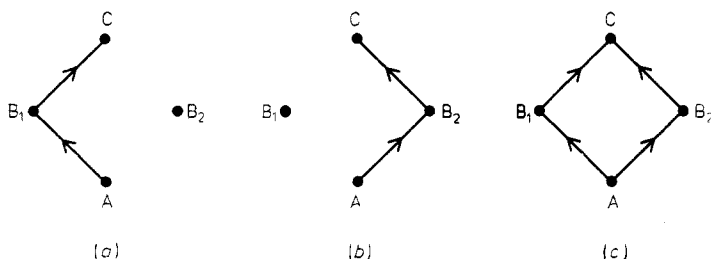


Figure 1. Diagrams contributing to $G(x, t+2; x, t)$. (a) and (b) give a contribution $+p^2$, while (c) gives $-p^4$.

These rules may be used to give a formal expression for G as follows. Define commuting operators $a(x, t)$ and $\bar{a}(x, t)$ on each site (x, t) , together with an operation Tr , which obey the algebra

$$\begin{aligned} a^2 &= ia & \bar{a}^2 &= i\bar{a} \\ \text{Tr } a &= \text{Tr } \bar{a} = 0 \\ \text{Tr } a\bar{a} &= 1. \end{aligned} \quad (1)$$

Then

$$G(x_2, t_2; x_1, t_1) = \text{Tr } a(x_2, t_2) \prod_{\substack{\text{links} \\ t' > t}} (1 + p\bar{a}(x', t')a(x, t))\bar{a}(x_1, t_1). \quad (2)$$

The central factor may be exponentiated in the form

$$\exp \lambda \sum_{x,t} \sum_{x',t'} \bar{a}(x', t') V(x' - x, t' - t) a(x, t) \quad (3)$$

where $\lambda = -\ln(1-p)$ and V depends on the lattice structure. In general V is short-ranged, even in $x' - x$, and vanishes for $t' \leq t$. The Fourier-Laplace transform of V is

$$V(k, E) = \int_{-\infty}^{\infty} d^D x \int_0^{\infty} dt \exp(ik \cdot x - Et) V(x, t) = c(1 - r_1 E - r_2 k^2 + \dots) \quad (4)$$

where c is the number of sites connected to a given site in the direction of increasing t , and r_1, r_2 are constants. Note that D is the number of transverse dimensions. Equation (3) is related to RFT by a Gaussian transformation. Introducing scalar fields $\psi(x, t)$, $\bar{\psi}(x, t)$ and taking the formal continuum limit

$$G = \text{Tr } a(x_2, t_2) \bar{a}(x_1, t_1) \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-\int dt d^D x (\bar{\psi}(\lambda V)^{-1} \psi + a\bar{\psi} + \bar{a}\psi)\right). \quad (5)$$

The operator V^{-1} has the expansion

$$V^{-1} = c^{-1}(1 + r_1 \partial_t - r_2 \nabla^2 + \dots). \quad (6)$$

Carrying out the operation Tr

$$G(x_2, t_2; x_1, t_1) = \sum_{m,n=1}^{\infty} \frac{(-i)^{m+n-2}}{m! n!} \langle \psi(x_2, t_2)^m \bar{\psi}(x_1, t_1)^n \rangle \quad (7)$$

where the correlation functions $\langle \psi^m \bar{\psi}^n \rangle$ are computed with the weight function $\exp(-A[\bar{\psi}, \psi])$ with

$$A = \int dt d^D x [(\lambda c)^{-1} (r_1 \bar{\psi} \partial_t \psi + r_2 \nabla \psi \cdot \nabla \bar{\psi} + \bar{\psi} \psi) - \bar{\psi} \psi + \frac{1}{2i} (\bar{\psi}^2 \psi + \bar{\psi} \psi^2)]. \quad (8)$$

In (8) we have dropped terms which are of higher order in $\bar{\psi}, \psi$, and higher-order derivatives, since they are irrelevant in $D = 4 - \epsilon$ transverse dimensions. A is just the action for RFT, and (7) is the elastic scattering cross section for high-energy particles in that theory. The only difference is that the various coupling constants, arbitrary in RFT, here depend on only p .

The percolation transition happens in mean field theory when the coefficient of $\bar{\psi} \psi$ vanishes, that is $p = 1 - e^{-1/c}$.

In RFT the renormalisation group has been used to obtain a scaling law for G in the critical region (Migdal *et al* 1974a,b, Abarbanel and Bronzan 1974a, b). One finds† (Abarbanel *et al* 1976)

$$G(x_2, t_2; x_1, t_1) \underset{\substack{t, x^2 \rightarrow \infty \\ p \rightarrow p_c}}{\sim} |p - p_c|^{\nu(\frac{1}{2}Dz - \eta)} \Phi(|p - p_c|^\nu t, |p - p_c|^{\nu z} x^2) \quad (9)$$

where $x = x_2 - x_1$ and $t = t_2 - t_1$. ν, η and z are critical exponents which depend only on D , and Φ is a calculable, universal scaling function which takes on different forms depending on whether p is greater than or less than p_c .

For $p < p_c$

$$G(x, t) \sim g^2 t^{-D/2} \exp(-x^2/4\alpha' t - \Delta t) \quad (10)$$

with

$$\begin{aligned} \Delta &\sim (p_c - p)^\nu \\ \alpha' &\sim (p_c - p)^{-\nu(z-1)} \\ g^2 &\sim (p_c - p)^{\nu[\frac{1}{2}D(z-1) - \eta]}. \end{aligned} \quad (11)$$

The 'susceptibility' is given by

$$\chi = \int dt d^D x G(x, t) \sim (p_c - p)^{-\gamma} \quad (12)$$

with $\gamma = \nu(1 + \eta)$.

For $p > p_c$

$$G(x, t) \sim M^2 \theta(vt - |x|) \quad (13)$$

with

$$v \sim (p - p_c)^{\nu(1 - \frac{1}{2}z)} \quad M \sim (p - p_c)^\beta \quad (14)$$

and $\beta = \frac{1}{2}\nu(\frac{1}{2}Dz - \eta)$. M is the matrix element of the field $\psi(x, t)$ between the two lowest energy states which become degenerate for $p \geq p_c$. In RFT the total cross section is proportional to M^2 , and in the percolation problem M gives the probability of a site belonging to an infinite cluster.

† We use the notation of RFT for the critical exponents, which is not quite uniform with that of statistical mechanics.

For $p = p_c$ (9) reduces to

$$G(\mathbf{x}, t) \sim t^{-(4Dz-\eta)} [\Phi_c(\mathbf{x}^2/t^z) + O(t^{-\lambda})]. \quad (15)$$

λ is another critical exponent describing the approach to scaling (Frazer and Moshe 1975). The scaling function Φ_c has been calculated in the one-loop approximation (Abarbanel *et al* 1975a, Frazer *et al* 1976).

Considerable effort has gone into the calculation of the critical exponents of RFT. The expansion in $\epsilon = 4 - D$ gives (Baker 1974, Bronzan and Dash 1974a, b, c)

$$\begin{aligned} \eta &= \frac{1}{12}\epsilon + \left(\frac{161}{12} \ln \frac{4}{3} + \frac{37}{24}\right) \left(\frac{1}{12}\epsilon\right)^2 + \dots \\ z &= 1 + \frac{1}{24}\epsilon + \left(\frac{59}{24} \ln \frac{4}{3} + \frac{79}{48}\right) \left(\frac{1}{12}\epsilon\right)^2 + \dots \\ \nu &= 1 + \frac{1}{12}\epsilon + \dots \\ \lambda &= \frac{1}{2}\epsilon + \dots \end{aligned} \quad (16)$$

These results are in agreement with the first-order ϵ calculation of Obukhov (1980) for the directed percolation problem, although Obukhov's definition of ν appears to be different from that given in (11). Direct calculations for $D = 2$ have been made using the loop expansion. They yield (Cardy 1976)

$$\eta = 0.26 \pm 0.02 \quad z = 1.13 \pm 0.01 \quad \lambda = 0.49 \pm 0.01. \quad (17)$$

The strong coupling expansion of a quantum spin model which is in the same universality class as RFT gives for $D = 2$ (Brower *et al* 1978)

$$\eta = 0.238 \pm 0.008 \quad z = 1.16 \pm 0.01 \quad \nu = 1.271 \pm 0.007, \quad (18)$$

and for $D = 1$

$$\eta = 0.317 \pm 0.002 \quad z = 1.272 \pm 0.007 \quad \nu = 1.736 \pm 0.001 \quad \lambda = 0.57 \pm 0.03. \quad (19)$$

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Note added in proof. After submitting this paper we learned of the work of Blease (1977a, b, c) and Kertész and Vicsek (1980). The series calculations of Blease are in excellent agreement with the RFT results quoted in this paper, using the scaling laws (12) and (14). The Monte Carlo result (Kertész and Vicsek 1980) that $\nu = 1.65 \pm 0.06$ for $D = 1$ is consistent, if somewhat low.

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